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Indeterminacy Problems in the Lisrel Model

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The latent variables and errors of the Lisrel model are indeterminate even when the parameters of the model are perfectly identified. The reason for the indeterminacy is that the Lisrel model gives a solution in terms of estimation of latent variables by means of observed variables. The *indeterminacy* is relevant also in practice; the minimum correlation between equivalent latent variables, is often negative in empirical examples. The degree of indeterminacy of the latent variables depends on the data. The average minimum correlation is a linear combination of the eigenvalues of the correlation matrix of solutions and it is always included in weak bounds which depend on the same eigenvalues.

In factor analysis two types of indeterminacy problems are considered. The first problem is known as the problem of identification of parameters of the model (Anderson & Rubin, 1956; Haagen 1983; Jöreskog, 1967; Lawley & Maxwell, 1963; Reiersöl, 1950); the second problem, the indeterminacy of factor scores, concerns the determination of factor scores (Guttman, 1955; Haagen, 1986; Heerman, 1964; Heerman, 1966; Kestelman, 1952; Ledermann, 1938; Piaggio, 1931; Schönemann, 1971; Schönemann & Haagen, 1987; Schönemann & Steiger, 1976; Schönemann & Steiger, 1978; Steiger & Schönemann, 1978; Schönemann & Wang, 1972; Steiger, 1979a, 1979b).

In this article we are dealing with the second problem in the Lisrel model (Jöreskog, 1970, 1977, 1978, 1981, 1982a, 1982b; Jöreskog & Sörbom, 1977; James, Mulaik, & Brett, 1982). We shall specify the necessary and sufficient conditions for the determination of the Lisrel model solutions. We shall then show some interesting results concerning the reasons for the indeterminacy of the Lisrel solutions. We define an index of indeterminacy and relate it to the covariance matrices of latent variables.

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The Lisrel Model

The Lisrel model considers vectors of random variables

$$\mathbf{z} = (z_1, z_2, \ldots, z_f)'; \mathbf{t} = (t_1, t_2, \ldots, t_g)'$$

of latent dependent and independent variables, respectively, in the following system of linear structural relations:

$$\mathbf{z} = \mathbf{L}_z\mathbf{z} + \mathbf{L}_t\mathbf{t} + \mathbf{p} \tag{1}$$

where **p** is a $f \times 1$ vector of residuals (errors in equations), $\mathbf{L}_t$ and $\mathbf{L}_z$ are $f \times g$ and $f \times f$ coefficient matrices, respectively ($\mathbf{L}_z$ has zeroes in the main diagonal and $\mathbf{I} - \mathbf{L}_z$ is nonsingular).

There are assumed linear relations between the vectors **z** and **t** and the observed vectors

$$\mathbf{x} = (x_1, x_2, \ldots, x_q)' \text{ and } \mathbf{y} = (y_1, y_2, \ldots, y_r)':$$

$$\mathbf{x} = \mathbf{L}_x\mathbf{z} + \mathbf{b} \tag{2}$$

$$\mathbf{y} = \mathbf{L}_y\mathbf{t} + \mathbf{h} \tag{3}$$

where $\mathbf{L}_x$ and $\mathbf{L}_y$ are $q \times f$ and $r \times g$ regression matrices of **x** on **z** and of **y** on **t**, respectively; **b** and **h** are $q \times 1$ and $r \times 1$ sets of errors of measurement (errors in variables) in **x** and **y**, respectively.

We assume that:

$$f \leq q; g \leq \rho \tag{4}$$

(5) $E(\mathbf{t}) = \mathbf{0}$; $E(\mathbf{p}) = \mathbf{0}$; $E(\mathbf{b}) = \mathbf{0}$; $E(\mathbf{h}) = \mathbf{0}$; $E(\mathbf{tp}') = \mathbf{0}$; $E(\mathbf{tb}') = \mathbf{0}$; $E(\mathbf{th}') = \mathbf{0}$; $E(\mathbf{pb}') = \mathbf{0}$; $E(\mathbf{ph}') = \mathbf{0}$; $E(\mathbf{bh}') = \mathbf{0}$.

Because we are dealing with the indeterminacy problem of latent variables and errors, we assume that the parameter matrices $\mathbf{L}_z, \mathbf{L}_t, \mathbf{L}_x, \mathbf{L}_y$, are known. Our problem is now how we can determine the scores of the latent variables having n independent observations of the random vectors **x** and **y**.

Let us indicate by **X** and **Y** the $q \times n$ and $r \times n$ matrices of observed scores of **x** and **y**; by **Z** and **T** the $f \times n$ and $g \times n$ score matrices of **z** and **t**; by **P** the $f \times n$ score matrix of **P**; by **B** and **H** the $q \times n$ and $r \times n$ score matrices of **b** and **h**, assuming the corresponding properties for the scores. The Lisrel model can be written:

$$\mathbf{Z} = \mathbf{L}_z\mathbf{Z} + \mathbf{L}_t\mathbf{T} + \mathbf{P} \tag{6}$$

$$\mathbf{X} = \mathbf{L}_x\mathbf{Z} + \mathbf{B} \tag{7}$$

$$\mathbf{Y} = \mathbf{L}_y\mathbf{P} + \mathbf{H}. \tag{8}$$

Let $\mathbf{S}_T, \mathbf{S}_P, \mathbf{S}_B, \mathbf{S}_H$ be the $f \times f$, $g \times g$, $q \times q$, $r \times r$ covariance matrices of **T**, **P**, **B**, **H** respectively, and:

$$\mathbf{L}_m = \begin{bmatrix} \mathbf{L}_x(\mathbf{I} - \mathbf{L}_z)^{-1}\mathbf{L}_t ; (\mathbf{I} - \mathbf{L}_z)^{-1} & ; \mathbf{I}_q & ; \mathbf{0} \\ \mathbf{L}_y \quad ; \quad \mathbf{0} & ; \mathbf{0} & ; \mathbf{I}_r \end{bmatrix}$$

$$\mathbf{S}_M = \begin{bmatrix} \mathbf{S}_T & & & \\ & \mathbf{S}_P & \mathbf{0} & \\ & \mathbf{0} & \mathbf{S}_B & \\ & & & \mathbf{S}_H \end{bmatrix}$$

with $\mathbf{S}_M$ an $m \times m$ matrix ($m = q + r + f + g$). Then it follows from the above assumptions that $\mathbf{S}_J$ is the $w \times w$ ($w = q + r$) covariance matrix of $\mathbf{J}$ [$\mathbf{J}' = (\mathbf{X}', \mathbf{Y}')$]:

$$\mathbf{S}_J = \mathbf{L}_m\mathbf{S}_M\mathbf{L}'_m. \tag{9}$$

Structure of Solutions and Sufficient Conditions

Conditions for Existence of Solutions

One can obtain solutions for the Lisrel model by determining the real value matrix:

$$\mathbf{M}' = \mathbf{T}', \mathbf{P}', \mathbf{B}', \mathbf{H}' \tag{10}$$

~~so that the relation follows:~~

$$\mathbf{J} = \mathbf{L}_m\mathbf{M} \tag{11}$$

depending on the condition that:

(12) $$\mathbf{MM'} = \mathbf{S}_M$$

with $\mathbf{L}_m$, $\mathbf{S}_M$ defined in Equation 9. We prove that we can always get a solution **M** satisfying Equations 11 and 12.

Theorem 1

Let $\mathbf{S}_J$ be the w x w covariance matrix of **J** of rank s ($0 < s \leq w$). If $\mathbf{L}_m$ is a real matrix of coefficient and $\mathbf{S}_M$ a gramian real matrix satisfying Equation 9, there exists a m x n matrix satisfying Equations 11 and 12.

Proof

(13) $$\mathrm{Rank}(\mathbf{S}_J) = \mathrm{Rank}(\mathbf{L}_m\mathbf{S}_M\mathbf{L'}_m) = \mathrm{Rank}(\mathbf{L}_m\mathbf{UU'L'}_m) \leq \mathrm{Rank}(\mathbf{U})$$

with **U** a m x u matrix. We assumed that $\mathrm{Rank}(\mathbf{S}_J) = s \leq w$. Hence:

(14) $$s \leq w \leq u$$

and the Euclidean vector space of w dimensions V_w formed by the w rows of **J** is contained in the u dimension of space V_u formed by the u columns of **U**. Every vector belonging to V_w can be obtained by a linear transformation of sets of vectors of V_u, in particular, of its orthonormal basis $\mathbf{U}_0$. Therefore, a w x f matrix **K** always exists:

(15) $$\mathbf{J} = \mathbf{KU}_0$$

and

(16) $$\mathbf{S}_J = \mathbf{KU}_0\mathbf{U'}_0\mathbf{K'} = \mathbf{KK'}$$

From Equations 13 and 16 it follows that $\mathbf{L}_m\mathbf{U}$ and **K** are both factor matrices of the same product matrix $\mathbf{S}_J$. For the properties of gramian matrices (Guttman 1944, p. 3) we know that there is an u x u orthogonal matrix $\mathbf{G}_u$ such that

(17) $$\mathbf{K} = \mathbf{L}_m\mathbf{UG}_u.$$

Hence, by Equation 16:

(18) $$\mathbf{S}_J = \mathbf{KU}_0\mathbf{U'}_0\mathbf{K'} = \mathbf{L}_m\mathbf{UG}_u\mathbf{U}_0\mathbf{U'}_0\mathbf{G'}_u\mathbf{U'L'}_m.$$

Putting:

(19) $$\mathbf{M} = \mathbf{UG}_u\mathbf{U}_0$$

results in

(20) $$\mathbf{J} = \mathbf{KU}_0 = \mathbf{L}_m\mathbf{M}$$

and

(21) $$\mathbf{S}_J = \mathbf{JJ'} = \mathbf{L}_m\mathbf{S}_M\mathbf{L'}_m = \mathbf{L}_m\mathbf{MM'L'}_m.$$

From the definition of $\mathbf{S}_M$ we deduce that the square matrices on the diagonal are composed respectively by g, f, q, r rows. Hence we can partition **M** into the g x n, f x n, q x n, r x n submatrices **T**, **P**, **B**, **H** respectively. Therefore a matrix **M** exists that satisfies Equation 11. Using Equations 11 and 21 becomes:

(22) $$\mathbf{S}_M = \begin{bmatrix} \mathbf{S}_T; & \mathbf{TP'}; & \mathbf{TB'}; & \mathbf{TH'} \\ \mathbf{PT'}; & \mathbf{S}_P; & \mathbf{PB'}; & \mathbf{PH'} \\ \mathbf{BT'}; & \mathbf{BP'}; & \mathbf{S}_B; & \mathbf{BH'} \\ \mathbf{HT'}; & \mathbf{HP'}; & \mathbf{HB'}; & \mathbf{S}_H \end{bmatrix}.$$

But, by hypothesis the matrix $\mathbf{S}_M$ is defined in Equation 9 and the matrices not set on the diagonal vanish. Therefore Equation 12 is satisfied.

Structure of Solutions: Sufficient Conditions

By Theorem 1, which proves the existence of solutions, we can conclude that generally there is not a unique solution. In fact, if the rank of $\mathbf{S}_M$ exceeds the rank of $\mathbf{S}_J$, the number of linearly independent vectors in **M** exceeds the number of linearly independent vectors in Equation 9, so that a unique solution cannot be determined even if the model is identified. So there are infinite matrices **M** having the same covariance matrix $\mathbf{S}_M$ and satisfying the conditions in Equation 11 and 12.

Theorem 2

Let **C** be an m x w matrix such that:

(23) $$\mathbf{CS}_j = \mathbf{S}_M\mathbf{L}'_m.$$

Let **W** be a m x u matrix such that:

(24) $$\mathbf{W} = (\mathbf{I} - \mathbf{RL}_m)\mathbf{U}$$

with **U** defined in Equation 13. Let **N** be an u x n orthonormal matrix such that:

(25) $$\mathbf{WNJ}' = \mathbf{0}.$$

A sufficient condition for **M** in Theorem 1 to satisfy Equation 11 and 12 is that **M** is of the form:

(26) $$\mathbf{M} = \mathbf{CJ} + \mathbf{WN}$$

CJ being a fixed component for a solution, **WN** being an indeterminate part.

Proof

Fixed Component for a Solution

CJ always exists and is unique. When $\mathbf{S}_j$ is nonsingular, **C** is uniquely determined:

(27) $$\mathbf{C} = \mathbf{S}_M\mathbf{L}'_m\mathbf{S}_j^{-1}.$$

When $\mathbf{S}_j$ is singular, by Equations 11 and 23 we have:

(28) $$\mathbf{MJ}' = \mathbf{S}_M\mathbf{L}'_m = \mathbf{CS}_j^{-1}$$

and therefore:

(29) $$(\mathbf{M} - \mathbf{CJ})\mathbf{J}' = \mathbf{0}.$$

Equation 29 is the set of normal equations for the least squares predictions of the variables in **M** from the variables in **J**, with **CJ** the *least square estimate*

and **WN** = **M** - **CJ** the matrix of errors of *estimate*. For **C** and $\mathbf{C}^*$, two solutions of Equation 29, we obtain:

(30) $$(\mathbf{C} - \mathbf{C}^*)\mathbf{S}_j = (\mathbf{C} - \mathbf{C}^*)\mathbf{JJ}' = \mathbf{0}$$

and:

(31) $$\mathbf{J} = \mathbf{E}_j\mathbf{G}_0$$

with $\mathbf{E}_j$ a w x w real matrix and $\mathbf{G}_0$ a w x n matrix whose rows are an orthonormal basis of space V_j. Hence in Equation 30:

(32) $$(\mathbf{C} - \mathbf{C}^*)\mathbf{E}_j\mathbf{E}'_j = \mathbf{0}$$

which means:

(33) $$(\mathbf{C} - \mathbf{C}^*)\mathbf{E}_j = \mathbf{0}.$$

Therefore:

(34) $$(\mathbf{C} - \mathbf{C}^*)\mathbf{E}_j\mathbf{G}_0 = \mathbf{0}$$

and by Equation 31, **CJ** is unique.

Indeterminate Part of a Solution

WN always exists and is not unique. It is always possible to define **N** as a matrix whose rows are an orthonormal basis of the space V_n, orthogonal to the space V_j. So **N** is an orthonormal matrix satisfying Equation 25. As there is more than one basis of the space, **WN** is not unique among all the solutions having same covariance matrix $\mathbf{S}_M$.

First Condition of Existence for a Solution

(35) $$\mathbf{L}_m\mathbf{M} = \mathbf{L}_m\mathbf{CJ} + \mathbf{L}_m\mathbf{WN} = \mathbf{J}$$

Hence, Equation 11 is satisfied from **M** defined in Theorem 2. In fact, by Equation 23 we obtain:

(36) $$(\mathbf{L}_m\mathbf{C} - \mathbf{I})\mathbf{S}_j = (\mathbf{L}_m\mathbf{C} - \mathbf{I})\mathbf{JJ}' = \mathbf{0}$$

and by Equation 36, proceeding as in Equation 30 for $(\mathbf{C} - \mathbf{C}^*)\mathbf{JJ}'$, we can state that $(\mathbf{L}_m\mathbf{C} - \mathbf{I})\mathbf{JJ}'(\mathbf{L}_m\mathbf{C} - \mathbf{I})'$ and $(\mathbf{L}_m\mathbf{C} - \mathbf{I})\mathbf{J}$ are equal to zero. Therefore $\mathbf{L}_m\mathbf{CJ} = \mathbf{J}$. Then, by Equations 23 and 24 we obtain:

$$\mathbf{L}_m\mathbf{WW}' = \mathbf{L}_m\mathbf{S}_M - \mathbf{S}_J\mathbf{C}' = \mathbf{0} \tag{37}$$

and by Equation 37, proceeding again as in equation 30 for $(\mathbf{C} - \mathbf{C}^*)\mathbf{JJ}'$, we can state that, also $\mathbf{L}_m\mathbf{WW}'\mathbf{L}_m'$, $\mathbf{L}_m\mathbf{W}$ and $\mathbf{L}_m\mathbf{WN}$ are equal to zero for any $\mathbf{N}$. Therefore Equation 35 is satisfied.

Second Condition of Existence for a Solution

$$\mathbf{L}_m\mathbf{MM}'\mathbf{L}'_m = \mathbf{L}_m(\mathbf{CJ} + \mathbf{WN})(\mathbf{CJ} + \mathbf{WN})'\mathbf{L}'_m = \mathbf{S}_J \tag{38}$$

and so Equation 12 is satisfied from M defined in Theorem 2. In fact by Equations 23 and 24 we obtain:

$$(\mathbf{L}_m\mathbf{J} + \mathbf{WN})(\mathbf{L}_m\mathbf{J} + \mathbf{WN})' = \mathbf{S}_M \tag{39}$$

and theorem 2 is established.

Structure of Solutions: Necessary Conditions

Theorem 3

In order to prove that always the solution of the model is of the form Equation 26, given Equations 22 to 24 it must result:

$$\mathbf{M} - \mathbf{CJ} = \mathbf{WN}. \tag{40}$$

Proof

By Equations 23 and 24 and the orthogonality of N, it follows that:

$$(\mathbf{M} - \mathbf{CJ})(\mathbf{M} - \mathbf{CJ})' = \mathbf{WW}'. \tag{41}$$

If any main diagonal element of $\mathbf{WW}'$ vanishes, the entire corresponding row of $(\mathbf{M} - \mathbf{CJ})$ must vanish too. Let $\mathbf{W}_c$ be the real submatrix of W nonvanishing rows of W, and $(\mathbf{M} - \mathbf{C})_c$ the corresponding nonvanishing rows of $(\mathbf{L} - \mathbf{CJ})$. We can then write:

$$(\mathbf{M} - \mathbf{CJ})_c(\mathbf{M} - \mathbf{CJ})_c = \mathbf{W}_c\mathbf{W}'_c. \tag{42}$$

By Theorem 1 (replacing J by $(\mathbf{M} - \mathbf{CJ})_c$, $\mathbf{L}_m$ by $\mathbf{W}_c$, $\mathbf{S}_M$ by I) a matrix of latent variables N always exists so that

$$(\mathbf{M} - \mathbf{CJ})_c = \mathbf{W}_c\mathbf{N}. \tag{43}$$

From Equation 43 we obtain:

$$(\mathbf{M} - \mathbf{CJ}) = \mathbf{WN} \tag{44}$$

and the necessary condition is proved.

In conclusion, Theorem 2 and 3 demonstrate that each solution of the Lisrel model is obtained by adding two components CJ and WN. The former is a least squares estimate of the latent variables in M from the observed variables in J, the latter is an error of estimation. The indeterminacy is caused by the fact that the *error of estimation* is in fact arbitrary.

A Measure of Indeterminacy

Minimum Average Correlation

Let M and $\mathbf{M}^*$ be two equivalent solutions such that

$$\mathbf{S}_J = \mathbf{L}_m\mathbf{S}_M\mathbf{L}'_m$$

and, by Theorem 1:

$$\mathbf{J} = \mathbf{L}_m\mathbf{M} = \mathbf{L}_m\mathbf{M}^* = \mathbf{L}_m(\mathbf{WN} + \mathbf{CN}) = \mathbf{L}_m(\mathbf{W}^*\mathbf{N}^* + \mathbf{CJ}). \tag{45}$$

The covariance matrix between the two solutions M and $\mathbf{M}^*$ is:

$$\mathbf{M}^*\mathbf{M}' = \mathbf{CS}_J\mathbf{C}' + \mathbf{W}^*\mathbf{N}^*\mathbf{N}'\mathbf{W}'. \tag{46}$$

The minimum of the trace of $\mathbf{M}^*\mathbf{M}'$ is the sum of the minimal covariances among elements of equivalent solutions. From Equation 45 we see that $\mathbf{W}^*\mathbf{N}^*$ and WN have the same norms for correspondent elements, and, for the Cauchy inequality, each main diagonal element of $\mathbf{W}^*\mathbf{N}^*\mathbf{N}'\mathbf{W}'$ cannot be larger than the correspondent element of $\mathbf{WNN}'\mathbf{W}'$. Therefore the minimum covariance between two solutions M and $\mathbf{M}^*$ is obtainable by putting:

(47) $$\mathbf{W^*N^*} = -\mathbf{WN}.$$

So we get in Equation 46:

(48) $$\min \operatorname{tr}(\mathbf{MM}^{*\prime}) = \operatorname{tr}(\mathbf{C}\mathbf{S}_j\mathbf{C}' - 2\mathbf{WW}')$$

and in terms of covariance and parameter matrices:

(49) $$\min \operatorname{tr}(\mathbf{MM}^{*\prime}) = \operatorname{tr}(2\mathbf{S}_M\mathbf{L}'_m\mathbf{S}_j^{-1}\mathbf{L}_m\mathbf{S}_M - \mathbf{S}_M).$$

The trace of the minimum correlation matrix:

(50) $$\min \operatorname{tr}(\mathbf{D}_M^{-1/2}\mathbf{MM}^{*\prime}\mathbf{D}_M^{-1/2}) = \min \operatorname{tr}(\mathbf{S}_M^*)$$
$$= \operatorname{tr}(2\mathbf{D}_M^{-1/2}\mathbf{S}_M\mathbf{L}'_m\mathbf{S}_j^{-1}\mathbf{L}_m\mathbf{S}_M\mathbf{D}_M^{-1/2}) - m$$

(with $\mathbf{D}_M$ diagonal matrix of variances of **M** and **M***) is the sum of the minimum correlation among corresponding latent variables of equivalent solutions.

Every element of the trace in Equation 50 is a normalized index of indeterminacy for the corresponding sets of latent variables or errors; the minimum average correlation (m.a.c.):

(51) $$\text{m.a.c.} = \frac{2\operatorname{tr}(\mathbf{D}_M^{-1/2}\mathbf{S}_M\mathbf{L}'_m\mathbf{S}_j^{-1}\mathbf{L}_m\mathbf{S}_M\mathbf{D}_M^{-1/2})}{m} - 1$$

is a normalized index of indeterminacy for all the latent variables of the Lisrel model.

Minimum Average Correlation and Eigenvalues of the Correlation Matrix

The researcher would be interested in a summary measure of indeterminacy whose validity does not depend on the structure of the data. However, the minimum average correlation points out how much indeterminacy is related to a particular structure of the model marked by particular matrices $\mathbf{L}_m$, $\mathbf{S}_j$, $\mathbf{S}_M$. Only if the latent variables and errors are standardized and mutually intercorrelated the minimum average correlation is always equal to $(q + r - f - g / m)$. Under the general assumptions of Equations 4 to 9 we can find a linear relation between the minimum average correlation and the eigenvalues of the correlation matrix of latent variables. By the property of the trace, Equation 50 is equivalent to:

(52) $$\min \operatorname{tr}(\mathbf{S}_M^*) = 2\operatorname{tr}(\mathbf{S}_j^{-1/2}\mathbf{L}_m\mathbf{D}_M^{-1/2}\mathbf{S}_M^{2*}\mathbf{D}_M^{-1/2}\mathbf{L}'_m\mathbf{S}_j^{-1/2}) - \operatorname{tr}(\mathbf{S}_M^*)$$

(with: $\mathbf{S}_M^* = \mathbf{D}_M^{-1/2}\mathbf{S}_M\mathbf{D}_M^{-1/2}$ correlation matrix of latent variables). By the properties of spectral decomposition of a symmetrical matrix:

(53) $$\mathbf{S}_M^{2*} = \sum_{i=1}^{m}(\mu_i\mathbf{v}'_i\mu_i\mathbf{v}_i)$$

with $\mathbf{v}'_i$ and μ_i *i*th eigenvector and eigenvalue of $\mathbf{S}_M^*$, respectively. Therefore we have in Equation 52:

(54) $$\min \operatorname{tr}(\mathbf{S}_M^*) = 2\operatorname{tr}[\mathbf{S}_j^{-1/2}\mathbf{L}_m\mathbf{D}_M^{-1/2}(\sum_{i=1}^{m}\mu_i\mathbf{v}'_i\mu_i\mathbf{v}_i)\mathbf{D}_M^{-1/2}\mathbf{L}'_m\mathbf{S}_j^{-1/2}] - \operatorname{tr}(\mathbf{S}_M^*).$$

By the properties of the eigenvalues the *k*th eigenvalue α_k of $\min(\mathbf{S}_M^*)$ is equal to:

(55) $$\alpha_k = \sum_{i=1}^{m}\mu_i(2\delta_{ik} - 1)$$

with δ_{ik} *k*th eigenvalue of the matrix

$$\mathbf{S}_j^{-1/2}\mathbf{L}_m\mathbf{D}_M^{-1/2}\mathbf{v}'_i\mu_i\mathbf{v}_i\mathbf{D}_M^{-1/2}\mathbf{L}'_m\mathbf{S}_j^{-1/2}.$$

By Equations 9 and 53 we have

(56) $$\sum_{i=1}^{m}\mathbf{L}_m\mathbf{D}_M^{-1/2}\mathbf{v}'_i\mu_i\mathbf{v}_i\mathbf{D}_M^{-1/2}\mathbf{L}'_m = \mathbf{S}_j$$

and therefore:

(57) $$\sum_{i=1}^{m}\delta_{ik} = 1.$$

In conclusion, the minimum average correlation of Equation 51 is a linear combination of the eigenvalues μ_i of the matrix $\mathbf{S}_M^*$:

$$\text{m.a.c.} = \sum_{i=1}^{m} \mu_i (2r_i - 1) \tag{58}$$

which weights r_i such as:

$$r_i = \mathrm{tr}\,(\mathbf{S}_j^{-1/2}\mathbf{L}_m\mathbf{D}_M^{-1/2}\mathbf{v}'_i\mu_i\mathbf{v}_i\mathbf{D}_M^{-1/2}\mathbf{L}'_m\mathbf{S}_j^{-1/2}) = \sum_{k=1}^{m} \delta_{ik};\ \sum_{i=1}^{m} r_i = m. \tag{59}$$

First Interval for the Average Minimum Correlation

By Equations 50 to 52 we know only that the average minimum correlation is bounded by -1 and 1. In order to obtain bounds for the minimum average correlation we can apply the methodology suggested by Schönemann and Wang (1972) for the factor model to the Lisrel model. For the properties of gramian matrices (Basilevsky 1983, p. 214) we have:

$$\mathbf{L}_m\mathbf{S}_M^{1/2} = \mathbf{R}^*\mathbf{G}_r \tag{60}$$

with $\mathbf{R}^* = [\mathbf{R},\mathbf{0}]$ a $w \times m$ matrix, $\mathbf{R}$ a $w \times w$ matrix and $\mathbf{G}_r$ an orthogonal $m \times m$ matrix. Therefore in Equation 50:

$$\min \mathrm{tr}(\mathbf{S}_M^*) = 2\mathrm{tr}[\mathbf{D}_M^{-1/2}\mathbf{S}_M^{1/2}\mathbf{G}'_r\mathbf{R}^{*\prime}(\mathbf{R}\mathbf{R}')^{-1}\mathbf{R}^*\mathbf{G}_r\mathbf{S}_M^{1/2}\mathbf{D}_M^{-1/2}] - m \tag{61}$$

and therefore:

$$\min \mathrm{tr}(\mathbf{S}_M^*) = 2\mathrm{tr}(\ \bar{\mathbf{I}}\mathbf{G}_r\mathbf{S}_M^{1/2}\mathbf{D}_M^{-1}\mathbf{S}_M^{1/2}\mathbf{G}'_r) - m \tag{62}$$

with:

$$\bar{\mathbf{I}} = \begin{bmatrix} \mathbf{I}_w & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

From the properties of the eigenvalues, the trace in Equation 62 is maximized (minimized) if the first w rows of the matrix $\mathbf{G}_r$ are the first (last) w eigenvectors of the matrix $\mathbf{S}_M^{1/2}\mathbf{D}_M^{-1}\mathbf{S}_M^{1/2}$. Therefore the min $\mathrm{tr}(\mathbf{S}_M^*)$ is always included in the following interval:

$$\sum_{i=w+1}^{m} \mu_i - m < \min \mathrm{tr}(\mathbf{S}_M^*) < \sum_{i=1}^{w} \mu_i - m \tag{63}$$

where μ_i is the ith eigenvalue of the matrix $\mathbf{S}_M^{1/2}\mathbf{D}_M^{-1}\mathbf{S}_M^{1/2}$, equal to the corresponding eigenvalue of the correlation matrix $\mathbf{S}_M^*$. By Equation 63 the average minimum correlation is always included in the following interval:

$$\frac{\sum_{i=w+1}^{m} \mu_i}{m} - 1 < \text{m.a.c.} < \frac{\sum_{i=1}^{w} \mu_i}{m} - 1 \tag{64}$$

but these bounds are weak because it is possible to choose the matrix $\mathbf{G}_r$ among all the $m \times m$ orthogonal matrices.

Second Interval for the Minimum Average Correlation

Let $\mathbf{L}_t$, $\mathbf{L}_p$, $\mathbf{L}_b$, $\mathbf{L}_h$ be four column blocks of the block matrix $\mathbf{L}_m$ defined in Equation 9. For the properties of the trace by Equation 50 we have:

$$\begin{aligned} \min \mathrm{tr}(\mathbf{S}_M^*) = 2\mathrm{tr}(&\mathbf{S}_j^{-1/2}\mathbf{L}_t\mathbf{S}_T\mathbf{D}_T^{-1}\mathbf{S}_T\mathbf{L}'_t\mathbf{S}_j^{-1/2} + \mathbf{S}_j^{-1/2}\mathbf{L}_p\mathbf{S}_P\mathbf{D}_P^{-1}\mathbf{S}_P\mathbf{L}'_p\mathbf{S}_j^{-1/2} \\ &+ \mathbf{S}_j^{-1/2}\mathbf{L}_b\mathbf{S}_B\mathbf{D}_B^{-1}\mathbf{S}_B\mathbf{L}'_b\mathbf{S}_j^{-1/2} + \mathbf{S}_j^{-1/2}\mathbf{L}_h\mathbf{S}_H\mathbf{D}_H^{-1}\mathbf{S}_H\mathbf{L}'_h\mathbf{S}_j^{-1/2}) - m. \end{aligned} \tag{65}$$

Once we have defined:

$\mathbf{t}'_i$ as the ith row of $\mathbf{S}_j^{-1/2}\mathbf{L}_t\mathbf{S}_T^{1/2}$;
$\mathbf{p}'_k$ as the kth row of $\mathbf{S}_j^{-1/2}\mathbf{L}_p\mathbf{S}_P^{1/2}$;
$\mathbf{b}'_l$ as the lth row of $\mathbf{S}_j^{-1/2}\mathbf{L}_b\mathbf{S}_B^{1/2}$;
$\mathbf{h}'_g$ as the gth row of $\mathbf{S}_j^{-1/2}\mathbf{L}_h\mathbf{S}_H^{1/2}$;

it follows from properties of the trace that Equation 65 is equivalent to:

$$\begin{aligned} \min \mathrm{tr}(\mathbf{S}_M^*) = {} & 2\sum_{i=1}^{m} \frac{\mathbf{t}'_i\mathbf{S}_T^{1/2}\mathbf{D}_T^{-1}\mathbf{S}_T^{1/2}\mathbf{t}_i}{\mathbf{t}'_i\mathbf{t}_i}\mathbf{t}'_i\mathbf{t}_i + 2\sum_{k=1}^{m} \frac{\mathbf{p}'_k\mathbf{S}_P\mathbf{D}_P^{-1}\mathbf{S}_P\mathbf{p}_k}{\mathbf{p}'_k\mathbf{p}_k}\mathbf{p}'_k\mathbf{p}_k \\ & + 2\sum_{l=1}^{m} \frac{\mathbf{b}'_l\mathbf{S}_B^{1/2}\mathbf{D}_B^{-1}\mathbf{S}_B^{1/2}\mathbf{b}_l}{\mathbf{b}'_l\mathbf{b}_l}\mathbf{b}'_l\mathbf{b}_l + 2\sum_{g=1}^{m} \frac{\mathbf{h}'_g\mathbf{S}_H^{1/2}\mathbf{D}_H^{-1}\mathbf{S}_H^{1/2}\mathbf{h}_g}{\mathbf{h}'_g\mathbf{h}_g}\mathbf{h}'_g\mathbf{h}_g - m. \end{aligned} \tag{66}$$

The maximum of Raleigh's quotients (Basilevsky 1983, p. 240) which appear in Equation 66 are maximized by the maximum eigenvalues $\tau_1^*, \pi_1^*, \beta_1^*, \theta_1^*$, of the matrices $\mathbf{S}_T^{1/2}\mathbf{D}_T^{-1}\mathbf{S}_T^{1/2}, \mathbf{S}_P^{1/2}\mathbf{D}_P^{-1}\mathbf{S}_P^{1/2}, \mathbf{S}_B^{1/2}\mathbf{D}_B^{-1}\mathbf{S}_B^{1/2}, \mathbf{S}_H^{1/2}\mathbf{D}_H^{-1}\mathbf{S}_H^{1/2}$, equal, for the properties of eigenvalues, to the maximum eigenvalues of the correlation matrices $\mathbf{S}_T^*, \mathbf{S}_P^*, \mathbf{S}_B^*, \mathbf{S}_H^*$. Therefore the trace in Equation 66 is inferior to:

$$(67) \qquad S_u = 2(\tau_1^* \sum_{i=1}^{m} \mathbf{t}'_i\mathbf{t}_i + \pi_1^* \sum_{k=1}^{m} \mathbf{p}'_k\mathbf{p}_k + \beta_1^* \sum_{l=1}^{m} \mathbf{b}'_l\mathbf{b}_l + \theta_1^* \sum_{g=1}^{m} \mathbf{h}'_g\mathbf{h}_g) - m.$$

Let $\mathbf{S}_{J1}$ and $\mathbf{S}_{J2}$ be the first q and the last r rows of the matrix $\mathbf{S}_J^{-1}$, respectively. For the structure of Lisrel in Equation 9 by Equation 67 we have:

$$(68) \quad S_u \leq 2[\,\bar{\tau}_1 \mathrm{tr}(\mathbf{I}_q - \mathbf{S}_B\mathbf{S}_{J1}) + \tau_1^* \mathrm{tr}(\mathbf{I}_r - \mathbf{S}_H\mathbf{S}_{J2}) + \beta_1^* \mathrm{tr}(\mathbf{S}_B\mathbf{S}_{J1}) + \theta_1^* \mathrm{tr}(\mathbf{S}_H\mathbf{S}_{J2})] - m$$

with $\bar{\tau}_1 = \max(\tau_1^*, \pi_1^*)$. Therefore the minimum average correlation is always inferior to:

$$(69) \qquad B_M = \frac{2[q\bar{\tau}_1 + r\tau_1^* + (\beta_1^* - \bar{\tau}_1)\mathrm{tr}(\mathbf{S}_B\mathbf{S}_{J1}) + (\theta_1^* - \tau_1^*)\mathrm{tr}(\mathbf{S}_H\mathbf{S}_{J2})]}{m} - 1.$$

By a similar process we can demonstrate that the minimum average correlation is always superior to:

$$(70) \quad b_m = \frac{2\,[q\,\bar{\tau}_{(f+g)} + r\,\tau_g^* + (\beta_q^* - \bar{\tau}_{(f+g)})\mathrm{tr}(\mathbf{S}_B\mathbf{S}_{J1})]}{m} + \frac{2\,[(\theta_r^* - \tau_g^*)\mathrm{tr}(\mathbf{S}_H\mathbf{S}_{J2})]}{m} - 1$$

where $\tau_g^*, \pi_f^*, \beta_q^*, \theta_r^*$ are the minimum eigenvalues of $\mathbf{S}_T^*, \mathbf{S}_P^*, \mathbf{S}_B^*, \mathbf{S}_H^*$, respectively and $\bar{\tau}_{(f+g)} = \min(\tau_f^*, \pi_g^*)$.

The bounds of Equations 69 and 70 are weak because we maximized and minimized the single elements of the trace in Equation 50.

Some Empirical Results

We have calculated the minimum correlation and the average minimum correlation for Lisrel's twelve models (or submodels) presented by Jöreskog (1982b). We assumed as in Kestelman (1952) and Schönemann and Wang (1972) that the models fit exactly into the sample, so that it follows that all the

conclusions drawn in the previous paragraph are valid. The solutions are calculated by the maximum likelihood method. We can see that the minimum correlation is always below one and is sometimes negative.

Conclusions

Even if the parameters of the Lisrel model are perfectly identifiable, there is not a unique solution for the latent variables of the model. This *indeterminacy* for the latent variables is caused by the fact that the Lisrel model gives a solution in terms of *estimation* of latent variables by means of observed variables.

In addition to noting the indeterminacy of scores on latent variables in the Lisrel model, James, Mulaik, and Brett (1982) note that indeterminacy of latent

Table 1
Minimum Correlation Coefficients

Models	Latent Variables				Errors					
1. Factor analysis 1^ (Jöreskog, 1982b)	.797	.788			.412	.479	.629	.812	.681	-.332
2. Factor analysis 2^ (Jöreskog, 1982a)	.645	.319			.383	.089	.416	.705	.796	.980
3. Variance analysis 1^ (Jöreskog, 1982b)	.534	.469			.694	.508	.731	.508	.731	.686
					.642	.642	.573	.344	.573	.344
4. Variance analysis 2^ (Jöreskog, 1982b)	.919	.297	.558	.817	-.210	.485	.782	.462	-.029	.259
					.189					
5. Simplex 1^ (Jöreskog & Sörbom, 1976)	.925	.180	.443	-.369	.327	.388	.490	.543	.288	.216
					.171					
6. Simplex 2^ (Jöreskog & Sörbom, 1976)	.744	-.430	-.140	-.676	.595	.487	.371	.554	.522	.575
					.533					
7. Simplex 3^ (Jöreskog & Sörbom, 1976)	.731	-.350	-.304	-.631	.340	.336	.216	.248	.319	.422
					.225					
8. Simplex 4^	.768	.916	.299	.248	.566	.701	.678	.744	.757	.570
(Jöreskog & Sörbom, 1976)	.001	.183	.245	-.128	.855	.720	.759	.724	.664	.629
				.784	.769	.819	.786	-.318	.874	
				.594	.840	.767				
9. Lisrel 1^ (Joreskog, 1973)	.584	.792	.722	-.045	.178	.433	.132	-.111	-.039	.305
	.073	.000	.000							
10. Lisrel 2^ (Jöreskog, 1982b)	.874	.889	.917	.703	.322	.399	-.071	.395	.667	.204
	.656				.512	.575	.583	-.305		
11. Lisrel 3^ (Jöreskog, 1982b)	.548	.462	.193		.528	.158	.418	.291	-.096	.683
12. Lisrel 4^ (Jöreskog, 1982b)	.541	.475	.204		.577	-.025	.469	.103	-.088	.703

Table 2
Average Minimum Correlation Coefficients

Models	Latent	Errors Variables	Total
1. Factor analysis 1^ (Jöreskog, 1982b)	.793	.448	.530
2. Factor analysis 2^ (Jöreskog, 1982b)	.482	.560	.540
3. Variance analysis 1^ (Jöreskog, 1982b)	.505	.581	.576
4. Variance analysis 2^ (Jöreskog, 1982b)	.647	.329	.435
5. Simplex 1^ (Jöreskog & Sörbom, 1976)	.294	.352	.332
6. Simplex 2^ (Jöreskog & Sörbom, 1976)	-.125	.541	.319
7. Simplex 3^ (Jöreskog & Sörbom, 1976)	-.138	.308	.159
8. Simplex 4^ (Jöreskog & Sörbom, 1976)	.316	.670	.581
9. Lisrel 1^ (Jöreskog, 1973)	.303	.032	.168
10. Lisrel 2^ (Jöreskog, 1982b)	.807	.284	.447
11. Lisrel 3^ (Jöreskog, 1982b)	.401	.330	.353
12. Lisrel 4^ (Jöreskog, 1982b)	.406	.289	.328

variables in Lisrel implies that while one may believe one confirms a model involving certain latent variables, one may actually be confirming the model because the manifest variables are determined by other variables than those hypothesized, which happen to have the same pattern of relationships to the manifest variables as given by one's hypothesis. While the likelihood of this happening is remote, its logical possibility implies one can never regard a structural hypothesis as true as opposed to *confirmed* because it is consistent with the data. This, James, Mulaik, and Brett say, is not a fatal flaw of latent variable models, but rather "... a reflection of the indeterminacy in all inductive attempts to use empirical observations to confirm theories by examining whether consequences deduced from theories do indeed occur as expected" (p. 166).

The *indeterminacy* is relevant also in practice: we saw that a normalized measure of indeterminacy, the minimum correlation between equivalent latent variables, is often negative in empirical examples.

Finally, the average minimum correlation between sets of correlated latent variables depends on the data. However it is possible to state that the average minimum correlation is a linear combination of the eigenvalues of the correlation matrix of the latent variables and it is always included in weak bounds which depend only on the same eigenvalues.

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